Metastability and Analyticity in a Dropletlike Model

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We consider an urn model closely related to the Fisher–Felderhof droplet model for the purpose of studying the relation between metastability and analytic continuation. For this model both the statics and dynamics can be solved and we confirm the relation between the metastable decay rate and the imaginary part of the analytically continued free energy (actually, pressure, in this model). We also find that eigenvalue degeneracy, an old theme for static aspects of phase transitions, appears in the dynamics as well. When approaching the phase transition from the stable side it is a degeneracy in the eigenvalues of the linear operator appearing in the master equation that causes the system to lock into a particular phase.

KEY WORDS: Metastability; droplet model; first-order phase transition.

1. INTRODUCTION

For first-order phase transitions several questions remain unanswered. First, one would like to know the nature of the singularity in the thermodynamic functions at the transition and the extent to which this singularity is model dependent. Second, there is the physically significant phenomenon of metastability which presents substantial problems within the mathematical framework (Penrose and Lebowitz,[1] Sewell[2]) of statistical mechanics with its emphasis on the thermodynamic limit. And finally, there is evidence of dramatic behavior within the metastable region. This comes under

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the rubric "spinodal," but until the second issue raised above is dealt with one hardly knows how to approach this last question.

Because the two-dimensional Ising model is unsolved for nonzero external magnetic field, past research on first-order transitions has used models with serious physical defects. Penrose and Lebowitz[3] found that a system with effectively infinite range forces could remain indefinitely in a state of nonminimal free energy (hence called metastable) and that, as for mean field theory, the thermodynamic functions were analytic at and beyond the first-order transition. On the other hand, the physically motivated droplet approximation (Andreev,[4] Fisher,[5] Langer[6]) suggested an "essential singularity," with deviations from analyticity of the form $\exp(-\mathrm{const}/H)$, where H is an external field that vanishes at the transition. Study of the droplet approximation was facilitated by the introduction of yet another idealized model for which that approximation is exact. In this one-dimensional model, which we call the Fisher–Felderhof model (Fisher,[5] Fisher and Felderhof,[7,8] Felderhof and Fisher,[9] Felderhof[10–12]) certain configurations of particles are designated "droplets." A droplet of size n is a maximal collection of n particles each of which has as neighbor at least one other member of the droplet within a specified range of it. An energy E_n is assigned to a size n droplet and the energy of the entire system is the sum of the individual droplet energies. The simplifying and unrealistic feature of this model is that the droplets do *not* interact with each other.

In this paper we further simplify the Fisher–Felderhof model, a step we were forced to take because of our interest in the dynamics of the transition. To motivate this concern with dynamics we now provide further background on conjectures that arose from the droplet approximation and from earlier studies that concentrated on the analytic properties of thermodynamic functions.

For a ferromagnetic system let $f(T, H)$ be the free energy per unit volume as a function of temperature and external field. As indicated earlier the analytic properties of f are problematic for first-order transitions. However, the function f can also be used to address the second question raised above. Thus, despite difficulties in defining the concept of metastable state, it may be that for $T < T_c$ ($=$ Curie temperature) the function f can be analytically continued from, say, the region $H \geqslant 0$, where it is obtained from a certain limit of the partition function, to $H < 0$. Since f is analytic with H a complex variable the singularity at $H = 0$ is no impediment to the analytic continuation unless, say, the imaginary H axis is a natural boundary. One would now like to relate the continued f to properties of laboratory metastable systems and this is where unresolved problems

of interpretation occur. To effect the continuation one can use various methods [droplet calculations (Langer,[6] Wallace[13]), transfer matrix (Newman and Schulman[14]), series expansions (Baker and Kim,[15] Privman and Schulman[16,17])].

In general, f which was real for real positive H, develops a nonzero imaginary part for real negative H. Quantitative comparison of the continued f with metastable dynamical systems is most easily done for model systems and in this way the derivative with respect to H of $\mathrm{Re}\,f$ has been compared to the magnetization in a dynamical Ising model. The system is put into a state in which most spins oppose the magnetic field and it rapidly relaxes to a well defined metastable state. Eventually the magnetization begins a sudden, rapid change and the system decays to the stable state. [This is the situation for low T and not too large a system; it arises presumably from the formation of a critical droplet. For very large systems and close to T_c critical droplets occur early on and the relaxation has a different character (Binder[18]). In this regime analytic continuation need have no relevance.] The correspondence of $\partial(\mathrm{Re}\,f)/\partial H$ with the mean magnetization prior to the rapid decay of the dynamical system is quite good over a large range of H (McCraw and Schulman[19]), supporting thereby the ideas on analytic continuation.

But $\mathrm{Im}\,f$ has presented a conceptual problem. Langer[20,21] studied the dynamics of the phase transition using the same droplet approach that he had used for the analytic continuation and found the decay rate per unit volume Γ to be proportional to $\mathrm{Im}\,f$ with the proportionality factor κ related to the properties of the critical droplet and nearby states. Comparison with experiment is not straightforward. Work of Huang et al.[22] appeared to allow metastability to persist well beyond the parameter values predicted by Langer, but contradiction was avoided (Binder and Stauffer,[23] Langer and Schwartz[24]) by taking into account critical slowing down and the time required for heat or other transport. The Monte Carlo studies of the dynamical Ising model cited above (McCraw and Schulman[19]) tend to confirm the $(\mathrm{Im}\,f) - \Gamma$ connection, but the precision is not good enough to make statements about κ. Finally, and this brings us close to the emphasis of the present paper, analytic studies of certain stochastic processes whose equilibrium properties exhibit a phase transition have been made (Newman and Schulman[25]). The systems are mean field models and the transition is the spinodal. Nevertheless, the decay rate through the spinodal is rigorously found and shown to be proportional to $\mathrm{Im}\,f$ with the factor κ correctly predicted by the droplet approximation— when appropriately interpreted. This correspondence holds only asymptotically near the phase transition, but this is quite reasonable. Many possible

dynamical laws reproduce the same equilibrium, but for finite distance into the metastable domain differences in the dynamical process will affect transition rates. One expects less ambiguity for Hamiltonian time evolution.

In the present paper we study a variant of the Fisher–Felderhof model which both exhibits a phase transition and to which can be assigned a tractable stochastic process. The model is flexible and is a laboratory for the ideas presented above. Implicit in our discussion was a mathematical framework for metastability, namely, analytically continue from the stable region (see Newman and Schulman[25] for further discussion). The continuation can be done and we once again validate the relation

$$\text{Im} \, f \sim \Gamma / \kappa \qquad (1.1)$$

Furthermore, we are able to study an exotic version of the model in which analytic continuation is impossible and there too we find corresponding trouble in the stochastic dynamics.

2. A CLASS OF MODEL SYSTEMS

Consider an urn of unit volume in contact with an infinite reservoir of particles held at temperature β^{-1} and chemical potential μ. If the urn contans n particles, it is taken to have energy E_n where $E_0 = 0$. Provided $n^{-1}E_n$ is bounded from below, the system attains equilibrium with pressure

$$p(\mu) = \beta^{-1} \log \sum_{n=0}^{\infty} e^{\beta(\mu n - E_n)} \qquad (2.1)$$

A model in this class is thus specified by prescribing the energies E_n. There is a connection between this class of urn models and a one-dimensional lattice gas related to the Fisher–Felderhof (FF) model [Gallavotti,[26] Roepstorff[27]]. In contrast to the urn models, the (lattice-) FF models have a nondenumerable set of states, i.e., the set of all subsets of the integers. To any $X \subset \mathbb{Z}$ interpreted as the set of occupied sites, one associates a translation invariant energy

$$U(X) = \sum_{Y \subset X} \phi(Y) \qquad (2.2)$$

where $\phi(X) = 0$ unless X is a cluster (i.e., an interval of integers) and if X is a cluster of size n, it is given the potential energy $J_n = \phi(X)$. The total energy is therefore

$$E_n = U(X) = \sum_{k=1}^{n} (n - k + 1) J_k \qquad (2.3)$$

As can be shown (Roepstorff[27]), the grand canonical ensemble leads to

the pressure

$$\bar{p}(\bar{\mu}) = p(\mu) \tag{2.4}$$

(barred quantities refer to the FF model) where $p(\mu)$ is given by (2.1) and the chemical potential of the FF model, $\bar{\mu}$, is related to μ:

$$\bar{\mu} = \mu + p(\mu) \tag{2.5}$$

This unfamiliar transformation of the chemical potential reveals its simple meaning in the canonical ensemble. From (2.4) and (2.5) we calculate the mean density of particles on the lattice as

$$\bar{\rho} = \frac{d\bar{p}}{d\bar{\mu}} = \frac{dp}{d\mu}\frac{d\mu}{d\bar{\mu}} = \frac{\rho}{1+\rho} \tag{2.6}$$

This shows how the specific volume, $\bar{v} = 1/\bar{\rho}$, of the particle on the lattice relates to the specific volume, $v = 1/\rho$, of a particle in the urn (at the same temperature and the same free energy per particle):

$$\bar{v} = v + 1 \tag{2.7}$$

Owing to the lattice spacing the minimal volume occupied by a particle is 1. To see that the free energies are indeed the same, we first determine the free energy per unit volume:

$$\bar{A}(\bar{\rho}) = \sup_{\bar{\mu}}\{\bar{\mu}\bar{\rho} - \bar{p}(\bar{\mu})\} = \sup_{\mu}\left\{(\mu + p(\mu))\frac{\rho}{1+\rho} - p(\mu)\right\}$$

$$= \frac{1}{1+\rho}\cdot\sup_{\mu}\{\mu\rho - p(\mu)\} = \frac{1}{1+\rho}A(\rho) \tag{2.8}$$

Then we pass to the free energy per particle, $F(v) = vA(v^{-1})$, and obtain the relation

$$\bar{F}(\bar{v}) = F(v) \tag{2.9}$$

Obviously, the macroscopic properties of the two models are the same apart from a trivial change of the specific volume.

We next introduce a stochastic dynamics for the lattice model and use the simplifying feature of the urn model to reduce the state space to a tractable size. The state at time t is specified by the number X_t of particles in the urn; X_t is a random variable taking values in $\{0, 1, 2, \ldots\}$. A Markov chain is obtained if, for discrete time steps, we fix the time-independent transition probabilities

$$p_{nm} = \text{Prob}(X_{t+1} = m \mid X_t = n) \tag{2.10}$$

$P = (p_{nm})$ is the transition matrix of the system. The constraints

$$0 \leqslant p_{nm} \leqslant 1, \qquad \sum_m p_{nm} = 1 \tag{2.11}$$

render P a stochastic matrix. Let $u_n(t) = \text{Prob}(X_t = n)$; then the time evolution of the probabilities u_n is governed by the master equation

$$u_m(t + 1) = \sum_n u_n(t) p_{nm} \tag{2.12}$$

The assumption of detailed balance is

$$w_n p_{nm} = w_m p_{mn} \tag{2.13}$$

for a set of numbers $w_n > 0$. Provided

$$Z = \sum_{n=0}^{\infty} w_n \tag{2.14}$$

exists, the numbers w_n yield the stationary state of the Markov process, $u_n = Z^{-1} w_n$. In the context of statistical mechanics, the w_n are Boltzmann weights defined by the system in equilibrium and Z is the partition function. For our model, the weights are taken to be

$$w_n = e^{\beta(\mu n - E_n)} \tag{2.15}$$

so that u_n is the Gibbs state of the urn.

There is ambiguity in choosing p_{nm}. One condition to reduce the freedom is to allow only jumps of magnitude unity:

$$p_{nm} = 0, \qquad |n - m| \geqslant 2 \tag{2.16}$$

This fixes P to be an infinite Jacobi matrix. However, even with the condition (2.16) there is still ambiguity in defining the Markov process. We take

$$p_{nm} = \alpha (w_m / w_n)^{1/2}, \qquad |n - m| = 1$$

$$p_{00} = 1 - p_{01} \tag{2.17}$$

$$p_{nn} = 1 - p_{nn+1} - p_{nn-1}, \qquad n \geqslant 1$$

where we choose $\alpha > 0$ small enough so that $p_{nn} \geqslant 0$ for all n. Such α can be found provided (2.15) holds and the difference $E_{n+1} - E_n$ is bounded from below.

Using matrix notation, we write

$$P = 1 - \alpha S^{-1} B S \tag{2.18}$$

where S is a diagonal matrix, $S_{nn} = (w_n)^{1/2}$. B is therefore positive and

symmetric:

$$
B = \begin{bmatrix} b_0 & -1 & & & \\ -1 & b_1 & -1 & & \\ & -1 & b_2 & -1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix},
\qquad
\begin{aligned}
b_0 &= (w_1/w_0)^{1/2} \\
b_n &= (w_{n+1}/w_n)^{1/2} + (w_{n-1}/w_n)^{1/2}
\end{aligned}
$$

$$(2.19)$$

From the assumption $E_{n+1} - E_n > C > -\infty$ we infer that the b_n's are bounded. Therefore, B may be viewed as a bounded linear operator on l^2, the space of square summable sequences, with spectrum contained in the interval $[0, 2/\alpha]$.

If $\sum w_n < \infty$, the vector

$$
u = \left[(w_0)^{1/2}, (w_1)^{1/2}, \dots \right]
\tag{2.20}
$$

belongs to l^2 and $Bu = 0$. Conversely, any vector v satisfying $Bv = 0$ is a multiple of u. Thus, stable systems are characterized by zero being an eigenvalue of B.

If B were a finite matrix, the remaining eigenvalues would be bounded away from zero and the state of equilibrium would be reached exponentially fast. This behavior is also characteristic for infinite systems away from the critical points where a phase transition occurs. However, if B is infinite and the chemical potential assumes its critical value, the remaining spectrum may extend down to zero as we shall see below. The disappearance of the spectral gap gives rise to a slowing down of the dynamical approach to equilibrium. This is how the dynamics reflect the phase transition.

3. THE LOGARITHMIC MODEL

We now turn to a specific model. Let

$$
E_n = \log n, \qquad n = 1, 2, \dots
\tag{3.1}
$$

The pressure is therefore

$$
p(\mu) = \begin{cases} \beta^{-1}\log\left(1 + \displaystyle\sum_{n=1}^{\infty} n^{-\beta} e^{\beta\mu n}\right), & \mu \leqslant 0 \\[2mm] \infty, & \mu > 0, \end{cases}
\tag{3.2}
$$

From the point of view of equilibrium statistical mechanics, the region

$\mu > 0$ is inaccessible, a feature reminiscent of the Bose gas; $\mu = 0$ is a critical value for the chemical potential of the urn. For the lattice, the corresponding value is

$$\bar{\mu}_c = p(0) = \beta^{-1} \log[1 + \zeta(\beta)] \qquad (3.3)$$

(ζ = Riemann zeta function) provided $\beta > 1$. For $\beta \leqslant 1$ the lattice model never reaches a critical state. At $\bar{\mu} = \bar{\mu}_c$ the system begins to condense. The condensation process is complete when all lattice points are occupied ($\bar{\rho} = 1$). Within the urn, this process is never completed: there is simply no state to which the system may tend; the density rises indefinitely.

As μ tends to zero (i.e., $\bar{\mu} \to \bar{\mu}_c$) we distinguish two regimes:

I. $1 < \beta \leqslant 2$:   The pressure remains finite, but the density $dp/d\mu$ tends to infinity.

II. $\beta > 2$:   Both the pressure and the density remain finite.

Technically speaking, the pressure is a convex function of the variable μ. As β exceeds 2, this function develops a cusp at $\mu = 0$. As a consequence, the derivative $dp/d\mu$ cannot be unique at $\mu = 0$. As is common in convex analysis, one introduces the concept of subdifferential to replace the derivative when it is not defined. We define the subdifferential of p at $\mu = 0$ by

$$\partial p(0) = \{\rho \,|\, p(\mu) \geqslant p(0) + \rho\mu\} \qquad (3.4)$$

and justify this notion by the following result: ρ minimizes the free energy, $A(\rho)$, if and only if $\rho \in \partial p(0)$.

Now $\rho \in \partial p(0)$ (we tacitly assume $\beta > 2$) is equivalent to $\rho \geqslant \rho_0$ where ρ_0 is the density of the urn at the onset of condensation

$$\rho_0 = \frac{\zeta(\beta - 1)}{1 + \zeta(\beta)} \qquad (3.5)$$

Initially, when we raise the density, the free energy $A(\rho)$ decreases until we reach the density ρ_0 to stay constant for $\rho \geqslant \rho_0$. This behavior of the free energy is reflected in the appearance of straight line portions of the isotherms, similar to those obtained in classical van der Waals theory by the Maxwell equal-area construction. It is an old belief that the "wiggle" in the van der Waals equation of state explains experimentally observed metastability. However, a contemporary view, allowing for a richer equation of state, generalizes the following of the wiggle to the analytical continuation of the free energy or pressure. In our model, this means continuing $p(\mu)$ to the domain Re $\mu > 0$. This is easily accomplished by using the Robinson[28] formula

$$\sum_{n=1}^{\infty} n^{-\beta} e^{\beta\mu n} = (-\beta\mu)^{\beta-1} \Gamma(1-\beta) + \sum_{n=0}^{\infty} (\beta\mu)^n \frac{\zeta(\beta - n)}{n!} \qquad (3.6)$$

($\beta \neq 1, 2, 3, \ldots$, Γ = gamma function). While the series on the left loses its meaning when Re $\mu > 0$, the series on the right converges for $\beta |\mu| < 2\pi$. If $\beta = 1, 2, 3, \ldots$, the equation (3.6) has to be replaced by

$$\sum_{n=1} n^{-\beta} e^{\beta \mu n} = \begin{cases} -\log(1 - e^{\mu}), & \beta = 1 \\[2mm] \dfrac{(\mu \beta)^{\beta - 1}}{(\beta - 1)!} \left[\displaystyle\sum_{n=1}^{\beta - 1} \frac{1}{n} - \log(-\beta \mu) \right] \\[4mm] \quad + \displaystyle\sum_{\substack{n=0 \\ n \neq \beta - 1}}^{\infty} (\beta \mu)^n \dfrac{\zeta(\beta - n)}{n!}, & \beta = 2, 3, \ldots \end{cases} \tag{3.7}$$

For all β, $\mu = 0$ is a branch point for $p(\mu)$. If we analytically continue around $\mu = 0$, the pressure acquires an imaginary part on the positive real axis with sign depending on the path around the origin. As μ tends to zero along the positive real axis,

$$\operatorname{Im} p(\mu) \to \begin{cases} \pm (\log \mu^{-1})^{-1}, & \beta = 1 \\[2mm] \pm a \cdot \mu^{\beta - 1}, & \beta > 1 \end{cases} \tag{3.8}$$

where

$$a = \pi \beta^{\beta - 2} \Gamma(\beta)^{-1} [1 + \zeta(\beta)]^{-1} \tag{3.9}$$

Let us now investigate the Markov process when μ changes sign. We take

$$w_0 = 1$$
$$w_n = n^{-\beta} e^{\beta \mu n} \tag{3.10}$$

and define the transition matrix by (2.18) and (2.19). To justify the use of Eq. (2.18) we observe that $E_{n+1} - E_n = \log(1 + 1/n)$ is bounded below. It is important that the process is well defined even if $\mu > 0$ where we hope to obtain information about the metastable decay rate.

For $\mu > 0$ the sum over the w_n's does not converge and thus the operator B loses its eigenvalue zero. Consequently, the urn's occupation grows indefinitely and never reaches equilibrium. However, for the associated lattice model there is the condensed state of unit density to which the system tends. We may say that in the regime $\mu > 0$ the Markov process models the condensation in the more realistic system.

If μ is positive, but small, and if $\beta > 1$, the Boltzmann weights w_n initially decrease with n, though they will ultimately tend to infinity. The significance of this fact is that for a long time states with small occupation do not notice the chemical potential's being nonzero. They will tend to a

near equilibrium, i.e., the metastable state, though the mean occupation grows slowly with time which we interpret as the decay of the metastable state.

Let us assume that the spectral gap of the operator B is the interval $[0,\lambda]$. Then the decay rate of the metastable state is

$$\Gamma = -\log(1 - \alpha\lambda) \tag{3.11}$$

as can be inferred from (2.18); λ is a function of $\mu > 0$ and is expected to tend to zero at a certain rate as $\mu \to 0$. The next section is devoted to the study of $\lambda(\mu)$, where we also want to compare Γ with $\text{Im}[p(\mu)]$.

Finally, we present a model which at first sight challenges the connection between metastability and analytic continuation. Suppose we alter the logarithmic model slightly so as to make the line $\text{Re}\,\mu = 0$ a natural boundary for the analytic function $p(\mu)$. Can metastability be observed in such a case? First we describe such an alteration and argue heuristically for an affirmative answer. However, in a later section we shall show the weakness of that heuristic argument. The argument runs as follows. With the radius of convergence of the series $\sum a_n z^{r_n}$ being 1, it can be shown that if $\lim r_n/n = \infty$, then $|z| = 1$ is the natural boundary of the analytic function defined by the series (Dienes[29]). For instance,

$$f(z) = z + z^2 + z^4 + z^8 + \cdots \tag{3.12}$$

is a typical "lacunary" series that cannot be continued to $|z| > 1$. The same is then true for

$$\sum_{n=0}^{\infty} z^n e^{-\beta E_n(\epsilon)} \equiv \sum_{n=0}^{\infty} z^n e^{-\beta E_n} + \epsilon f(z), \qquad \epsilon \geqslant 0 \tag{3.13}$$

We take

$$p(\mu,\epsilon) = \beta^{-1}\log \sum_{n=0}^{\infty} e^{\beta[\mu n - E_n(\epsilon)]} \tag{3.14}$$

as the pressure of a fictitious model. The argument for dynamical metastability runs as follows: Since ϵ may be chosen as small as we please, the physics of the system will not be noticeably different from that of the system where $\epsilon = 0$. Furthermore, we may assume $p(\mu,0)$ to be defined for $\text{Re}\,\mu < 0$ and continuable to $\text{Re}\,\mu > 0$ where it acquires an imaginary part on the positive real axis. On the other hand, for no $\epsilon > 0$, can $p(\mu,\epsilon)$ be continued beyond $\text{Re}\,\mu = 0$. This argument, an apparent contradiction to the underlying ideas of metastability, will be analyzed more carefully below.

4. STOCHASTIC DYNAMICS FOR THE URN

As stated above the urn is in contact with a heat and particle reservoir at temperature T ($= 1/\beta$) and chemical potential μ. The state at time $t \in Z$ is specified by the random variable X_t, the number of particles in the urn. On a given time step, X_t can change by unity with transition probability $p_{ij} = \text{Prob}(X_{t+1} = j \mid X_t = i)$. The properties of p_{ij} are given in Eqs. (2.11)–(2.17) above. Recalling the notation of Section 2, the equilibrium state of the system is

$$u_n(t) = \text{Prob}(X_t = n) = w_n \Big/ \sum_{k=0}^{\infty} w_k \qquad \text{(for all time)}$$

We recall our particular choice of p_{ij}

$$p_{ij} = \alpha \left(\frac{w_j}{w_i} \right)^{1/2} \qquad \text{for } |i - j| = 1 \qquad i, j \geqslant 0 \qquad (4.1)$$

$$p_{ii} = 1 - p_{i\,i+1} - p_{i\,i-1}, \qquad i \geqslant 1 \qquad (4.2)$$

$$p_{00} = 1 - p_{01} \qquad (4.3)$$

where α is a fixed positive number small enough so that $p_{ii} > 0$ for all i. Another common choice for p_{ij} is to replace Eq. (4.1) by $p_{i\,i+1} = \alpha w_{i+1}/w_i$, $p_{i\,i-1} = \alpha$. One can also select p_{ij} so that all diagonal elements but p_{00} vanish, which has the advantage of reducing the arbitrariness of the choice. For convenience in handling the matrices, however, we stick to Eqs. (4.1)–(4.3).

The $\{w_j\}$ for the logarithmic model are

$$w_0 = 1$$
$$w_n = n^{-\beta} \exp(\beta \mu n), \qquad n \geqslant 1 \qquad (4.4)$$

which for $\mu < 0$ establishes the equilibrium state of the stochastic process to be the Gibbs state of the urn.

Our interest is in $\mu = 0$ and $\mu > 0$. At $\mu = 0$, which for $\beta > \beta_c = 1$ is a first-order phase transition, it will be instructive to see how the transition is reflected in the dynamics. (In fact, we shall see that eigenvalue degeneracy, useful for understanding long-range order, also plays a role in halting the wandering of the state in time.) For $\mu > 0$ the sum over the w_i's does not converge and the urn's occupation grows indefinitely. However, as we have seen above the urn free energy can be analytically continued to $\text{Re } \mu > 0$ and we now examine the rate at which the urn begins its headlong filling. Now despite the nonexistence of an asymptotic equilibrium state this system is similar to the usual metastable decay process in that along the

way to the distant asymptotic condition there is a barrier. The process is considerably slowed at the barrier and until it gets past it the system will have properties not significantly different from nearby $\mu < 0$ states. Specifically, the transition matrix p_{ij} will have nearly equal values for increase or decrease of urn occupation for that k at which the function $w_k = k^{-\beta}\exp(\beta\mu k)$ is stationary. Near such k the process looks like diffusion *without* drift and progress is slow. The minimum of w_k for small positive μ is at $k_0 \sim 1/\mu$. For $k < k_0$ there is drift toward smaller k so that the very reaching of k_0 is unlikely. For $k > k_0$ there is drift toward higher k so that once the system passes k_0, it is well on its way. The foregoing is analogous to the role of the critical droplet in metastability.

For $\mu \geqslant 0$ it is convenient to truncate the stochastic process. From the above discussion of the $\mu > 0$ case, it is clear that we will have full knowledge of the rate at which systems find their way to the regime of rapid growth if we truncate the stochastic process at some N with $N \gg 1/\mu$. This truncation is accomplished by setting the transition probability $N + 1 \rightarrow N$ to zero. With this modification, we can, in the master equation, Eq. (2.12), confine attention to u_n, $n \leqslant N$, since once an urn has occupancy greater than N it never turns back. Therefore it is sufficient to consider the $(N + 1) \times (N + 1)$ matrix P_N whose matrix elements are just the p_{ij}, $i, j = 0, \ldots, N$. This is not a stochastic matrix since the sum of the elements of its last row is not unity. For the case $\mu = 0$ we make a slightly different change in the stochastic process: for some selected N, systems which would under the old scheme have made the transition $N \rightarrow N + 1$ are now held at N. Operationally this means we consider an $(N + 1) \times (N + 1)$ transition matrix Q_N whose matrix elements are identical to those of P_N except that

$$(Q_N)_{NN} = 1 - p_{N\,N-1} \tag{4.5}$$

[For P_N we had $(P_N)_{NN} = 1 - p_{N\,N-1} - p_{N\,N+1} = p_{NN}$.] Q_N is therefore a stochastic matrix with largest eigenvalue 1. That is, it has an equilibrium state which in fact is the left eigenvector $(w_0, w_1, \ldots, w_N)$.

At a first-order phase transition one is used to the idea of there being two (or more) competing states of the system. However, the time for passage from one of these states to the other should be a rapidly increasing function of system size. For the master equation this situation could be reflected in a near degeneracy of the largest eigenvalue so that initial amplitudes along the eigenvector of next largest eigenvalue would decay extremely slowly. Thus for the Ising model on an $M \times M$ lattice the Gibbs state has its support on two classes of configurations widely separated in terms of single spin flips. This suggests a degeneracy on the order of $\exp(-\text{const} \cdot M^2)$. The usual macroscopic time scale in Ising simulations is only M^2 spin flip attempts per second so that relaxation from one "phase"

to the other would be exponentially long in the volume. For the urn system our basic singularity is a branch point rather than an exponential. Nevertheless, we shall still find that at the first-order transition relaxation is slowed and the recovery time grows as N^2. Specifically we will show the degeneracy behaves as $1/N^2$ with N the value at which we modify the stochastic process. By contrast, with μ a fixed small negative number the degeneracy is of order unity so that any reasonable macroscopic time scale would be $o(N)/\text{sec}$ (with N again a cutoff). Thus at the first-order transition recovery will go like

$$\left(1 - \frac{\text{const}}{N^2}\right)^{o(N)} \to 1$$

which is to say the system will remain in the nonequilibrium state.

To study the matrices P_N and Q_N it is convenient to perform a similarity transformation with the $(N + 1) \times (N + 1)$ matrix $S = \text{diag}[(w_i)^{1/2}]$. The new matrices are Hermitian and their spectra unchanged. For $SQ_N S^{-1}$ the eigenvector with eigenvalue 1 is $[(w_0)^{1/2}, (w_1)^{1/2}, \ldots, (w_N)^{1/2}]$. It is also convenient to eliminate the portions of P_N and Q_N proportional to the identity. Let

$$A = \frac{1}{\alpha} S[1 - Q_N] S^{-1}, \qquad B = \frac{1}{\alpha} S[1 - P_N] S^{-1} \qquad (4.6)$$

(N dependence suppressed). It follows that the eigenvalues of A are equal to or greater than zero, and those of B strictly greater [cf. Eq. (2.18) and discussion following]. Thus we study the following questions: (I) What is A's first eigenvalue exceeding 0? And (II) what is B's lowest eigenvalue?

(I) Let λ be the smallest eigenvalue of A other than 0. The eigenvector for the eigenvalue 0 is $\tilde{w} \equiv [(w_0)^{1/2}, (w_1)^{1/2}, \ldots, (w_N)^{1/2}]$, so λ can be characterized through the constrained variational problem:

$$\lambda = \inf_\xi \left(\frac{\xi^\dagger A \xi}{\xi^\dagger \xi} \right), \qquad \text{where} \quad \xi^\dagger \xi > 0 \quad \text{and} \quad \xi^\dagger \tilde{w} = 0$$

Therefore we have an upper bound on λ: For any $x \neq 0$ with $x^\dagger \tilde{w} = 0$

$$\lambda < \frac{x^\dagger A x}{x^\dagger x}$$

We now exhibit an $O(1/N^2)$ bound. Consider a test vector $x_k = \phi_k(w_k)^{1/2}$. We shall not give an exact specification of ϕ_k, but will characterize it by several properties. ϕ_k will be slowly varying (and of order unity) so that x is *nearly* the eigenvector $\tilde{w}$; however, ϕ_k will have a change in sign (at least one "node" in continuum language) to allow orthogonality: $x^\dagger \tilde{w} = \sum \phi_k w_k = 0$. Also we require $\phi_k = 0$ for $k = 0$ and N to avoid certain endpoint contributions.

Then $x^\dagger x = \sum \phi_k^2 w_k$, which is some finite number as $N \to \infty$ for $\beta > 1$. It is easy to see that

$$x^\dagger A x = \sum_k \phi_k (w_k)^{1/2} \left\{ \left[\left(\frac{w_{k+1}}{w_k} \right)^{1/2} + \left(\frac{w_{k-1}}{w_k} \right)^{1/2} \right] \phi_k (w_k)^{1/2} \right.$$

$$\left. - \phi_{k+1}(w_{k+1})^{1/2} - \phi_{k-1}(w_{k-1})^{1/2} \right\}$$

$$= \sum_k \phi_k (w_k)^{1/2} \left[\phi_k (w_{k+1})^{1/2} + \phi_k (w_{k-1})^{1/2} \right.$$

$$\left. - \phi_{k+1}(w_{k+1})^{1/2} - \phi_{k-1}(w_{k-1})^{1/2} \right]$$

$$= \text{end point contributions} \quad (\text{at } k = 0 \text{ and } N)$$

$$+ \sum (w_k w_{k-1})^{1/2} (\phi_k - \phi_{k-1})^2 \tag{4.7}$$

The end point contributions (which disappear because $\phi_0 = \phi_N = 0$) came from changing the summation index from k to $k + 1$ for the first and third terms in the brackets. Now ϕ_k makes changes of order 1 in the course of N steps; therefore $|\phi_k - \phi_{k-1}| = O(1/N)$ so that

$$x^\dagger A x = \sum (w_k w_{k-1})^{1/2} [O(1/N)]^2 = O(1/N^2)$$

$$x^\dagger x = O(1)$$

Hence $\lambda \leqslant O(1/N^2)$.

It turns out that $\lambda = O(1/N^2)$ is not just an upper bound, but is in fact the actual dependence. We shall demonstrate this by both numerical and analytical means. Define f_k to be the determinant of the $(k + 1) \times (k + 1)$ matrix consisting of the first $k + 1$ rows and columns of $A - \lambda I$. Then λ satisfies $f_N = 0$. Defining f_{-1} to be 1, it is easy to see that

$$f_k = (a_k - \lambda) f_{k-1} - f_{k-2} \tag{4.8}$$

with a_k the diagonal matrix element of A,

$$a_k = \begin{cases} (w_1)^{1/2}, & k = 0 \\[2mm] \left(\dfrac{w_{k+1}}{w_k} \right)^{1/2} + \left(\dfrac{w_{k-1}}{w_k} \right)^{1/2}, & 1 \leqslant k \leqslant N - 1 \\[2mm] \left(\dfrac{w_{N-1}}{w_N} \right)^{1/2}, & k = N \end{cases} \tag{4.9}$$

Now $\lambda = 0$ is always a solution, but our interest is in the smallest $\lambda > 0$ for which $f_N = 0$. In Table I we present numerical results for $2 \leqslant N \leqslant 300$,

Table I. N, λ **(the Smallest Positive Eigenvalue)**
and $N^2\lambda$ for the $(N + 1) \times (N + 1)$ Matrix A

N	λ	$N^2\lambda$
2	1.8938 92775	7.5755 71100
10	0.18646 04938	18.646 04938
40	0.01232 40653	19.718 50451
80	0.00311 66517	19.946 57080
100	0.00199 94097	19.994 09702
120	0.00139 07042	20.026 14112
200	0.00050 22775	20.091 09897
300	0.00022 36002	20.124 01513

for $\beta = 4$. Note that a factor N^2 makes λ essentially constant. By the method of rational extrapolation using $N \geqslant 80$ we obtain $\lim_{N \to \infty} N^2\lambda_N$ $= 20.1907281$.

With this numerical demonstration of the N^2 dependence of λ we derive analytically the actual asymptotic value of N^2 as a root of a Bessel function. Rewrite Eq. (4.8) as

$$-f_{k+1} + 2f_k - f_{k-1} + (a_{k+1} - 2 - \lambda)f_k = 0 \qquad (4.10)$$

Using the expansion

$$\left(\frac{w_{k\pm 1}}{w_k} \right)^{1/2}$$

$$= e^{\pm \beta\mu/2}\left[1 \mp \frac{1}{2} \beta \frac{1}{k} + \frac{1}{2} \beta\left(\frac{1}{2} \beta + 1 \right) \frac{1}{2k^2} \right.$$

$$\left. \mp \frac{1}{2} \beta\left(\frac{1}{2} \beta + 1 \right)\left(\frac{1}{2} \beta + 2 \right) \frac{1}{3!k^3} + O(k^{-4}) \right] \quad (4.11)$$

for $\mu = 0$ we see that a_k is close to 2 for large k different from N. Define $f(x) = f_k$ for $x = k/N$ and $l^2 = \lambda N^2$. Then (4.10) becomes for $\mu = 0$ and $\nu = (\beta + 1)/2$

$$\frac{d^2f}{dx^2} + \left(\nu^2 - \frac{1}{4} \right) \frac{1}{x^2} f = l^2 f \qquad (4.12)$$

with errors of order k^{-2}. At $k = N$, a_k has only one term of the form (4.11) and to lowest order (4.10) is

$$\frac{1}{2} \beta f(1) + \frac{df}{dx}\bigg|_{x=1} = 0 \qquad (4.13)$$

Equations (4.12) and (4.13) do not fix l^2 since the small x behavior of f is unspecified and is somewhat more delicate an issue because the continuum

approximation does not hold there. If it turns out that the $x = 0$ boundary condition is N independent then we will in fact have shown that l^2 is asymptotically independent of N and therefore $\lambda \sim \text{const}/N^2$.

One solution to Eq. (4.12) is $f(x) = x^{-\beta/2}$ with $l = 0$; this reflects the known invariant state $u_k = (w_k)^{1/2}$. In general (4.12) admits two sorts of behavior near $x = 0$, $f \sim x^{-\beta/2}$ or $f \sim x^{1+\beta/2}$. Solutions that vanish near zero may be expected to reflect solutions of the matrix equation since where the continuum approximation breaks down the solution is nearly zero anyway. In fact (4.12) is essentially the equation for the spherical Bessel function and is solved by

$$f(x) = \sqrt{x}\, J_\nu(lx)$$

Moreover, the boundary condition (4.13) can be rewritten as

$$\frac{d}{dx}\left[x^{\beta/2}f(x)\right] = 0 \qquad \text{for} \quad x = 1$$

which, with the help of the recursion formulas for the Bessel function, allows us to state the condition on l as

$$J_{\nu-1}(l) = 0 \tag{4.14}$$

For the case $\beta = 4$, $J_{3/2}(l) = 0$ is equivalent to $l = \tan l$ and l^2 to ten significant figures is 20.190 72856, in excellent agreement with the numerical value given above.

Although we have not supplied a rigorous proof it is apparent that the lowest eigenvalue of A above zero is given by l^2/N^2 where l satisfies (4.14). The function f is also the continuum limit of the associated eigenvector and is the ground state of the "Schrödinger equation" (4.12) with boundary conditions (4.13) and $f'(0) = 0$. The $x = 0$ boundary condition for this solution differs significantly from that of the solution $x^{-\beta/2}$. Apparently for finite large N the corresponding eigenvectors manage to be orthogonal through small negative contributions near zero from the eigenvector that goes on to be a Bessel function.

By contrast, with $\mu < 0$ the lowest eigenvalue above zero is $O(1)$, independent of N so that the natural time scale for the urn is 1 time step/sec.

(II) We next turn to the matrix B, which governs the metastable decay rate. Because P_N is not stochastic, B does not have the eigenvector $[(w_0)^{1/2}, (w_1)^{1/2}, \dots]$ and in contrast to A, it is the lowest eigenvalue of B that governs the metastable decay (see Newman and Schulman[25]). As for A we can use the variational principle and calling the lowest eigenvalue of B β_0 we have

$$\beta_0 \leqslant \frac{x^\dagger B x}{x^\dagger x} \tag{4.15}$$

with no requirement on x other than not being identically zero. As a test vector take $x_j = \phi_j(w_j)^{1/2}$, as we did earlier. However, we do not require $\phi_0 = 0$ and on the contrary take $\phi_0 = 1$. We will still demand that ϕ_j tend to zero for sufficiently large j (and in fact expect it to vanish fairly rapidly for j well beyond $1/\mu$). The algebra involved in Eq. (4.7) goes through unchanged and with the demand that ϕ_N actually be zero we have

$$\beta_0 \leqslant \inf_\phi \frac{\sum_{k=1}^N (w_k w_{k-1})^{1/2}(\phi_k - \phi_{k-1})^2}{\sum_{k=1}^N \phi_k^2 w_k} \equiv \inf_\phi \frac{\gamma[\phi]}{\nu[\phi]} \qquad (4.16)$$

First let us consider this expression heuristically, ignoring the normalization denominator $\nu[\phi]$. One could, as for the matrix A, estimate β_0 by taking $\phi_k = 1$ for $k \leqslant 1/\mu$ and $\phi_k = 0$ thereafter. For the usual droplet model situations softening this cutoff changes nothing since the droplet surface energy is relatively less singular than droplet volume as, say, magnetic field approaches zero. But now consider the sum in γ. The quantity w_k [essentially the same as $(w_k w_{k-1})^{1/2}$] starts out $O(1)$ for k near 0, drops to $O(\mu^\beta)$ for k near $1/\mu$ and then grows large again as k increases further. Obviously we want to spread the changes in ϕ_k over the region $k \sim 1/\mu$ and the question is how broad is that region. Since w_k is minimum at $k \sim \bar{k} \equiv 1/\mu$, we can write

$$w_k \sim w_{\bar{k}} \exp\left[(k - \bar{k})^2/2\Delta^2\right]$$

Recalling that $w_k = k^{-\beta} \exp(k\mu\beta)$ and calculating $\partial^2 w_k/\partial k^2$ at $\bar{k}$ yields

$$w_{\bar{k}} = e^\beta \mu^\beta, \qquad \Delta = 1/\mu$$

Therefore the region of change for ϕ is $O(1/\mu)$ so that $|\phi_k - \phi_{k-1}|$ is of order μ. Therefore the sum in γ has $1/\mu$ significant terms, each of order $w_k \mu^2$, and

$$\beta_0 \leqslant \frac{1}{\mu} w_{\bar{k}} \mu^2 \sim \mu^{\beta+1} \qquad (4.17)$$

More rigorous justification of (4.17) is achieved by recognizing the numerator $\gamma[\phi]$ as a kind of action for a system with time-dependent mass $[L \sim \frac{1}{2} m(t)\dot{x}(t)^2, \; t \leftrightarrow k, \; m \leftrightarrow (w_k w_{k-1})^{1/2}, \; \phi \leftrightarrow x]$. The extremum is achieved by taking $m\dot{x} = \text{const}$, which translates to

$$(w_k w_{k-1})^{1/2}(\phi_k - \phi_{k-1}) = -\delta = \text{const}$$

For this fixed δ the total sum γ is

$$\gamma = \sum (w_k w_{k-1})^{1/2}(\phi_k - \phi_{k-1})^2 = \sum \frac{\delta^2}{(w_k w_{k-1})^{1/2}}$$

But δ is set by the normalization of ϕ:

$$1 = \sum_{n=1}^{N} (-\phi_k + \phi_{k-1}) = \delta \sum \frac{1}{(w_k w_{k-1})^{1/2}}$$

Collecting terms we have

$$\beta_0 \sim \gamma, \qquad \gamma^{-1} = \sum \left[\frac{1}{(w_k w_{k-1})^{1/2}} \right] \sim \sum_{k=1}^{N} k^{\beta} e^{-\beta \mu k}$$

By Laplace's method this is asymptotically (for $\mu \downarrow 0$)

$$\gamma \sim \mu^{\beta+1} \sqrt{\beta}\, e^{\beta}$$

The denominator $\nu[\phi] = \sum \phi_k^2 w_k$ is a nonsingular function of μ since ϕ_k goes to very small values before w_k begins to make trouble. Therefore this sum is estimated by

$$\sum_{k=1}^{1/\mu} w_k = \sum_{k=1}^{1/\mu} k^{-\beta} e^{\beta \mu k}$$

whose main contribution is from k near 1 and which is therefore nearly independent of μ for small μ. It follows therefore that

$$\beta_0 \gtrsim \mu^{\beta+1}$$

Time evolution under the matrix P_N does not conserve probability. If we imagine an initial probability distribution to be expanded in terms of the eigenfunctions of P_N then the asymptotic loss of probability is governed by the largest eigenvalue of P_N. Writing this largest eigenvalue as $e^{-\Gamma}$, Γ is the decay rate with time measured in time steps of the stochastic process. By the definition of B and for small β_0 we have

$$\Gamma = -\log(1 - \alpha\beta_0) \sim \alpha\beta_0 \qquad (4.18)$$

We have thus established a decay rate $\Gamma \sim \mu^{\beta+1}$, where we now assume that the upper bound given above is not just a bound, but is actually attained. Evidence for that assumption is presented in Table II.

In Section 3 of this paper we used the work of Robinson[28] to show that $\mathrm{Im}[p(\mu)] \sim \mu^{\beta-1}$. The fact that Γ does not share this dependence is not a disagreement with the Langer theory (Langer[20,21]) since there is the need to insert a kinetic factor which can indeed be μ dependent. Specifically, Langer finds

$$\Gamma \sim \kappa\, \mathrm{Im}(\text{free energy}) = \kappa\, \mathrm{Im}\big[p(\mu) \big] \qquad (4.19)$$

with κ a kinetic factor which we now calculate.

In contrast to our calculations up to this point the determination of κ depends on a specific model.

Table II.   A List of the Smallest Eigenvalue for Various β and μ for the $N \times N$ Matrix B. For Each β We Wish to Establish the Form $\lambda \sim \mu^m$ and Fit the Value of m. For Successive Values of λ (λ' and λ'') We Calculate $\tilde{m} = [\log(\lambda'/\lambda'')]/[\log(\mu'/\mu'')]$. This is Listed as Well as its Extrapolation to $\mu = 0$.

β	μ	λ	$\tilde{m}$	$\tilde{m}_0$
3	0.2	$5.894E - 3$	3.6332	
3	0.1	$4.750E - 4$	3.8111	
3	0.05	$3.384E - 5$	3.9075	
3	0.025	$2.256E - 6$		4.009
4	0.4	$6.203E - 2$	4.0717	
4	0.2	$3.689E - 3$	4.5515	
4	0.1	$1.574E - 4$	4.7973	
4	0.05	$5.657E - 6$	4.9082	
4	0.025	$1.884E - 7$		5.006
6	0.4	$5.224E - 2$	5.5346	
6	0.2	$1.127E - 3$	6.3748	
6	0.1	$1.358E - 5$	6.7326	
6	0.05	$1.277E - 7$		7.049

The decay of the metastable state is pictured as passing through a critical droplet. Although we justified this for the urn and although we used this to motivate the choice of x in estimating β_0, we have not until now had to adopt this model. From Langer[21] κ is the eigenvalue of the most unstable mode in the neighborhood of the critical droplet. Specifically the free energy is expanded in a quadratic form about the droplet state. In most directions in function space this droplet state is a local minimum and the quadratic form has positive eigenvalues. For systems with translation invariance (which the urn is not) there will be corresponding zero eigenvalues. In the direction along which the droplet is unstable the free energy is maximal and the associated eigenvalue of the quadratic form is just κ. For the urn model the droplet "state" corresponds to an urn having occupation $k_0 \sim 1/\mu$. The "free energy" that one uses in the droplet calculation is the microscopic free energy which would in fact be the energy if one were describing states with exact microscopic coordinates. For the urn this is just $-(1/\beta)\log w_k$ with w_k as usual $k^{-\beta}\exp(\beta\mu k)$. The second derivative of $\log w_k$ with respect to k is $(\text{const})/k^2$ so that $\kappa \sim \mu^2$. This yields $\kappa \operatorname{Im} p \sim \mu^{\beta+1}$ in agreement with the $\mu^{\beta+1}$ dependence of Γ derived above.

Finally we reexamine the system mentioned at the end of Section 3. For $f(z)$ given by a lacunary power series with radius of convergence 1 the

urn energies are defined as in Eq. (3.13). In that section we took it to be "obvious" that although analytic continuation to $\mu > 0$ ($|z| > 1$) was impossible the stochastic process was substantially unchanged for small enough ϵ.

As observed earlier, the relation $\Gamma/\kappa = \text{Im } p$ cannot be expected to be rigorously true for μ finitely away from zero for systems with stochastic dynamics. This is because a slight change in spin flip rules can affect the precise value of Γ/κ for the finite droplets. Such a change, however, has no influence on the equilibrium pressure $p(\mu)$, nor on its analytic continuation. Therefore the best that can be hoped for is an asymptotic relation $\Gamma/\kappa \sim \text{Im } p$ for $\mu \downarrow 0$. But for fixed ϵ there will always be a sufficiently small μ for which the dynamics are in fact affected. For $\mu \downarrow 0$ the size of the matrix B used for studying the decay rate grows as some multiple of $1/\mu$. (This is clear from our estimates and has been confirmed numerically as well.) Therefore no matter how sparse the series for $f(z)$, eventually there will be an arbitrarily large number of terms from f in the range $0 \leqslant k \leqslant O(1/\mu)$. It is thus clear that the influence of $f(z)$ can be important, but let us see in detail how the eigenvalues of the matrices A and B are affected. In our demonstrations regarding B we used the quantities w_k, in a way that was indifferent to whether or not a given w_k contains contributions from f. The important estimate involved a quantity $(\gamma^{-1})_{\min} \sim \sum (w_k w_{k-1})^{-1/2}$. However, for sufficiently small μ this sum is in fact determined by f. For the usual w_k's the minimum term occurs for $k \sim 1/\mu$, for which $w_{1/\mu} > \mu^\beta e^\beta$ a quantity smaller than ϵ for small enough μ. Therefore although there might even (for appropriate f) exist a limit of λ for $\mu \downarrow 0$ its properties would be determined neither by $f(e^{\beta \mu})$ nor by $p(\mu)$ for $\mu < 0$.

For the matrix A [for which the limit $(-\mu) \downarrow 0$ may be more appropriate] slightly different trouble erupts because of f. Recalling Eq. (4.9) and those following, the important properties of λ arose because a_k for large k was approximately equal to 2. Now (with f) that breaks down.

5. SUMMARY AND CONCLUSIONS

We have studied a model exhibiting a first-order phase transition from both a dynamical and thermodynamical point of view. Our goal was to check the relationship of the concepts of metastability and analytic continuation and in particular the formula $\text{Im}[p(\mu)] \sim \Gamma/\kappa$ where $p(\mu)$ is the pressure as a function of chemical potential, Γ is the decay rate out of the metastable state under the stochastic dynamics, and κ is a kinetic factor for that decay. The imaginary part of $p(\mu)$ may be nonzero because μ is in a metastable regime and $p(\mu)$ is an analytic continuation of the real, physical $p(\mu)$ defined in the stable regime. The model is an "urn" model in which an energy is associated with having a number of particles in the urn and a

stochastic dynamics is defined consistent with the thermodynamics of the urn, namely, its equilibrium state is the Gibbs state of the urn. The urn model is closely related to a lattice droplet model whose thermodynamics had been studied earlier (Roepstorff[27]).

The chief message of this paper is that the proposed relation between metastability and analytic continuation is verified both for the logarithmic model and for our exotic, singular variation on it. We found this to be impressive since our investigations at first showed disagreement even for the logarithmic model. But then as subtle properties of the matrix associated with the decay rate calculation began to emerge all the factors came together and conspired to confirm the relation $\mathrm{Im}\, p \sim \Gamma/\kappa$. The second confirmation arose from what we at first considered a counterexample. The urn model is sufficiently flexible to produce a system for which analytic continuation is impossible. At first it seemed that dynamical metastability should be unaffected and that if the metastability-analytic continuation connection would be retained then somehow these pathological urn models would have to be disqualified on "higher" grounds, invoking nature's simplicity perhaps. However, higher grounds were not needed and we find in fact that when the pressure (or free energy) has a natural boundary there is good reason to expect trouble with the dynamical process as well.

Another point that we make in this paper is a simple observation concerning the relation $\mathrm{Im}\, p \sim \Gamma/\kappa$. The fact that many rules for stochastic dynamics lead to the same equilibrium state implies that for a finite distance into the metastable regime the above relation may not hold exactly. Hence the best that can be expected is an asymptotic relation at the transition. Hamiltonian dynamics, however, may allow for a sharper correspondence.

Finally, the old theme of eigenvalue degeneracy at phase transitions appears here in a new and interesting way: in the *dynamics*. The dynamics of the system are studied by means of a master equation and the eigenvalues of the associated linear operator. As the system approaches the first-order transition from the stable side (so this observation does not depend on any ideas about metastability) the eigenvalue associated with equilibrium (namely, 1) is approached by another eigenvalue sufficiently rapidly to cause the system's equilibration to become extremely slow. This is not the critical slowing down of second-order phase transitions although it certainly suggests a way to study that problem too.

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